LETTER TO THE EDITOR

Vertex models and directed walks: length scale exponents from finite size scaling

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Abstract. By using the mapping between ferroelectric five-vertex (or the modified KDP) model and directed walks, we prove, via finite size scaling, that the length scale exponents $\nu_\parallel$ and $\nu_\perp$ are related by $\nu_\parallel = 2\nu_\perp$ in all dimensions.

The ferroelectric five-vertex (or the modified KDP) model (5VM) (figure 1(a)) is one of the simplest models in the class of exactly solvable two-dimensional vertex models [1, 2]. Attempts have recently been made to study them in higher dimensions by defining the model on appropriate lattices as, e.g., the diamond lattice in its 110 orientation [3]. Such a generalization is warranted not only for complete understanding of these models, but also because of the usefulness, mainly through the equivalent directed walk (DRW) representation (figure 1(b)) [1, 3], in various contexts such as the commensurate-incommensurate (CI) (Pokrovsky-Talapov) transition [5] biomembrane phase transition [6-9], polymers in random media [10], flux lattice melting in high T_c superconductors [12], anyons [11], dimer models [1, 13] etc. The question of perennial interest is 'what are the thermal length scale exponents for 5VM?'

It is known from the exact finite size scaling analysis [14] and correlation functions [15] of the isomorphic Kasteleyn dimer model [9, 13] that the model is anisotropic, requiring two length scale exponents $\nu_\parallel$ parallel to the ground state polarization direction or the special z direction for the directed walk, and $\nu_\perp$ for the transverse direction, with $\nu_\parallel = 1$ and $\nu_\perp = \frac{1}{2}$. A simple minded scaling analysis was used in [3] to propose that these numbers are independent of dimensions. Here we prove by an essential use of the finite size scaling (FSS) theory [16] that $\nu_\parallel = 2\nu_\perp$ for all dimensions.
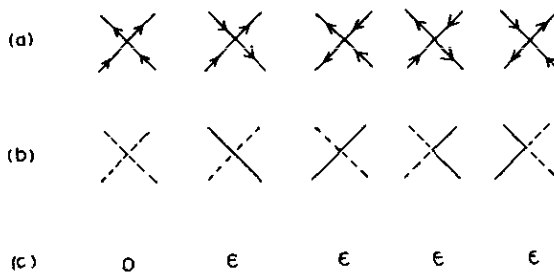
(a)

(b)

(c)   o   ϵ   ϵ   ϵ   ϵ

Figure 1. (a) The arrow configurations for the five vertex model (5VM), (b) the equivalent line configurations with the thick (dashed) line representing the presence (absence) of a line, (c) the energy of the vertex.

We use the DRW analogy in our analysis. In fact, DRWs appear implicitly in Lieb's solution of the general six-vertex models [2] via the Bethe ansatz, though the method cannot be generalized to higher dimensions. In contrast, our continuum path integral approach is in principle valid for all dimensions though at the cost of foregoing the underlying lattice structure, not so great a price if universlity is to be trusted. We point out only the relevant features here, referring the reader to the literature for details [1, 3, 7, 8, 9]. (i) The nth excited state of SVM consists of n mutually avoiding directed walks so that the temperature or $t \equiv (T - T_c)/T_c$ determines the density ρ of the walks (per unit tranverse volume), (ii) the loss in entropy due to the non-overlapping constraint is responsible for the nontrivial critical behaviour, (iii) the length of each walk is the size of the lattice in the z direction and (iv) the density is related to t through the incommensuration exponent $\bar{\beta}$,

$$\rho \sim t^{\bar{\beta}} \tag{1}$$

with $\bar{\beta} = 1/2$ in two dimensions.

The continuum model used in [3] and [4], in analogy with the flux lattice melting theory [12], is

$$\mathcal{H}_n = \frac{1}{2} \sum_{\alpha=1}^{n} \int_0^N dz \left(\frac{\partial \rho_\alpha(z)}{\partial z} \right)^2 + v_0 \sum_{\alpha < \beta} \int_0^N dz\, \delta^{d-1}[\mathbf{r}_\alpha(z) - \mathbf{r}_\beta(z)] \tag{2}$$

where $\mathbf{r}_\alpha(z)$ is the $d'(=d-1)$-dimensional co-ordinate of the point at contour length z of chain α each of length N, n being the total number of chains. The first term on the RHS of (2) is the usual entropic contribution (or the elastic energy) of the polymers, while the second term ensures the mutual repulsion at the same z of any pair of chains α, β. It is to be noted that, in this formulation, the z direction gets a special treatment compared to the remaining $d-1$ coordinates, reflecting the anisotropy of the system. Also, a simple dimensional analysis identifies the upper critical dimension as $d = 3$ where v_0 is dimensionless [3].

As pointed out already, t determines the density of the walks, playing the role of chemical potential for the polymer system. We however prefer to use the canonical ensemble with fixed n and fixing the density at the end. We show elsewhere [4] that the free energy of SVM at a temperature t is equal to the osmotic pressure of the directed chains with the correct choice of ρ as in (1). However, this connection is not crucial for this letter.

The crux of the analysis lies in the recognition that the scaling properties of the polymers [17] in the limit of large length N is, in the case at hand, the finite size scaling form of SVM as the system approaches the thermodynamic limit. According to FSS, any quantity, say the free energy, should show a scaling form of the type

$$f_N(t) \sim N^{-(2-\alpha)/\nu_\parallel} \mathcal{F}(t N^{1/\nu_\parallel}) \tag{3}$$

for small t because the correlation length in the z direction goes like $\xi_\parallel \sim t^{-\nu_\parallel}$ [14, 16, 18]. The important point to observe is the appearance of the combination variable $t N^{1/\nu_\parallel}$ in the argument of the scaling function. The connection with the polymer problem is through this.

For polymers each of length N, the osmotic pressure would have been like a perfect gas if the chains were non-interacting. But for large chains the interaction cannot be ignored and the osmotic pressure Π is expressible in a scaling form through a reference density above which overlaps are expected (similar to c^* for conventional polymers [17]). This is the density where the average size of a chain is comparable to the average

distance between the chains, and is determined quantitatively by the second virial coefficient A_2 [17], so that the osmotic pressure can be written as

$$\frac{\Pi}{kT\rho} = \mathscr{P}(A_2\rho). \tag{4}$$

In the limit $N \to \infty$, $A_2 \sim N^\psi$, with a new exponent ψ, so that the scaled density variable is

$$\bar{\rho} \equiv \rho N^\psi. \tag{5}$$

For the problem in hand, it is the same N that appears in (3) and (5), in one case (3) determining the 'rounding' of the critical behaviour, and in the other case (4) the density-dependent crossover from dilute to semidilute regime. Since the density ρ of (4) is related to the temperature of (3) by the incommensuration exponent $\bar{\beta}$ of (1), we obtain by comparing the combination· variables of both cases

$$\nu_\parallel = \bar{\beta}/\psi. \tag{6}$$

Therefore, the thermodynamic exponent $\bar{\beta}$ and the virial coefficient exponent ψ determine the length scale exponent $\nu_\parallel$ in the z direction.

The length scale exponent in the transverse direction is the average separation of the lines and is therefore determined by (1)·as

$$\nu_\perp = \frac{\bar{\beta}}{d-1} \tag{7}$$

since the density is defined per unit transverse volume.

The second virial coefficient for directed walks has been determined in [19] through a renormalization group (RG) analysis and it is shown† that, for $d < 3$, at the RG fixed point for $N \to \infty$, (see also [20])

$$A_{2R} = V(4\pi N)^{(d-1)/2} \frac{\sin \pi\varepsilon/2}{\pi(1 - \varepsilon/2)} \tag{8}$$

giving $\psi = (d-1)/2$. Using this value in (6), we finally get

$$\nu_\parallel = 2\nu_\perp \tag{9}$$

as we were set to prove.

To get the actual values of the exponents, one really needs $\bar{\beta}$, which is beyond the scope of this letter as it requires the calculation of the osmotic pressure of the many chain system. The details are to be published elsewhere [4]; we just quote the final result $\bar{\beta} = (d-1)/2$ as found in [3]. This gives $\nu_\parallel = 1$ and $\nu_\perp = \frac{1}{2}$ for all $d < 3$.‡

A few comments are in order.

(1) FSS is based on the hypothesis (or fact) that the length of the system does not require any renormalization in an RG approach [16]. It is, therefore, reassuring that the length N of the directed chains do ·not require any renormalization [19], giving further credence to the approach of this paper.

(2) It is known [22] that FSS in the form (3) is not valid for $d \geq$ the upper critical dimension, in this case three. This is easily seen for $d = 3$ because the virial coefficient develops a log correction, obtained by solving the exact beta function of [19] for $\varepsilon = 0$. This log term vitiates the connection through FSS, but still if one ignores it, the power laws are consistent with the above exponents for $\nu_\parallel$ and $\nu_\perp$. For $d > 3$, the exponents are generally the same as at the upper critical dimension, and therefore, we expect (9) to be true for all d.

† Note that d of this paper is $d+1$ of [19].
‡ In an approximate calculation for the flux lattice problem of the high T_c superconductors, Ziegler [21] also obtained $\nu_\parallel = 1$ and $\nu_\perp = 1/2$ in *three* dimensions. ₂

(3) The factor of two in equation (9) has a simple explanation through the 'roughness' exponent ν that describes the wandering or size of a chain as $N \to \infty$, $R \sim N^{\nu}$ (as measured, say, by the mean square end-to-end distance or by the radius of gyration). Now, the second virial coefficient of a hard sphere gas goes as the volume of the sphere, and therefore for the DRWs we expect $A_2 \sim R^{d-1}$, giving $\psi = (d-1)\nu$. Using the random walk value, as is applicable for us, $\nu = 1/2$, we recover the exponent of (8). This also shows that the general form of (9) for non-Gaussian walks, as may be relevant for random lattices, is $\nu\nu_{\parallel} = \nu_{\perp}$ a form already suggested by a scaling analysis [3].

(4) The reason behind the superuniversality of the relation (9) lies in the absence of any anomalous dimension for the second virial coefficient. This can further be traced to the absence of any self-interaction of the chains, unlike conventional polymers [17], so that the exponents are determined purely from dimensional analysis as established in [19]. This is also a noteworthy feature that comes out of our analysis and not so easy to see from the available exact solutions.

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According to [3] the generating function of the self-avoiding walks defined in equation (1) can be expressed as

$$C_L(x) = \lim_{N \to 0} N^{-1} \langle S(0, 0) \cdot S(L, L) \rangle. \tag{5}$$

The quantity on the right denotes the correlation function of spins on opposite corners of a square of $(L+1)^2$ classical N-component spins, normalized so that $S \cdot S = N$, with nearest-neighbour interactions. Since $T \to T_c$ in the magnetic system corresponds to $x \to \mu^{-1}$ in the system of self-avoiding walks [3], the result $x^* = \mu^{-1}$, suggested as a strong possibility in [1], follows directly from this correspondence.

Table 1. Total number of walks $C_L(1)$ between opposite corners of the square and between the midpoints of opposite edges.

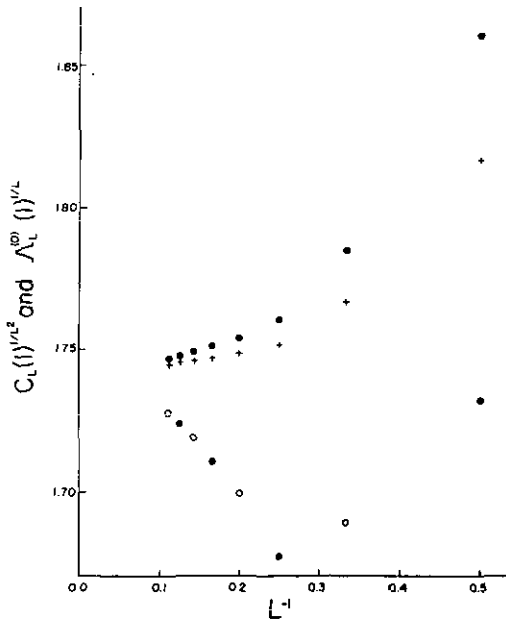| L	$(0, 0) \to (L, L)$	$(0, L/2) \to (L, L/2)$	$(0, (L-1)/2) \to (L, (L+1)/2)$
1	2		2
2	12	9	
3	184		112
4	8 512	3 915	
5	1 262 816		574 068
6	575 780 564	247 484 661	
7	789 360 053 252		338 670 045 504
8	3 266 598 486 981 642	1 378 292 310 954 861	
9	41 044 208 702 632 496 804		17 160 258 555 040 648 616

Figure 1. Dependence of $C_L(1)^{1/L^2}$ (circles) and $\Lambda_L^{(0)}(1)^{1/L}$ (crosses) on L^{-1}, with $L = 2, 3, \ldots, 9$. The upper and lower sequences of circles correspond to walks between opposite corners and the midpoints of opposite edges, respectively. The filled and empty points of the lower sequence correspond to even and odd L.